Physics 606 Exam 1 Solution

1. (a) $\boxed{\langle x \rangle =} \langle n | x | n \rangle$ with $x = \frac{1}{2b}(a + a^\dagger)$, $b = \left(\frac{m\omega}{2\hbar}\right)^{1/2}$

$$= \frac{1}{2b}\left(\langle n|a|n\rangle + \langle n|a^\dagger|n\rangle\right)$$

$$= \frac{1}{2b}\left(\sqrt{n}\,\underbrace{\langle n|n-1\rangle}_{=0} + \sqrt{n+1}\,\underbrace{\langle n|n+1\rangle}_{=0}\right)$$

$$= \boxed{0}$$

(b) $\boxed{\langle p \rangle =} \langle n | p | n \rangle$ with $p = \frac{1}{2c}(a - a^\dagger)$, $c \equiv \frac{(2m\hbar\omega)^{1/2}}{2i}$

$$= \frac{1}{2c}\left(\langle n|a|n\rangle - \langle n|a^\dagger|n\rangle\right)$$

$= \boxed{0}$ as in part (a)

(c) Since $\langle p \rangle = 0$ & $\langle x \rangle = 0$, $\boxed{\Delta p = \langle p^2 \rangle^{1/2} \text{ \& } \Delta x = \langle x^2 \rangle^{1/2}}$.

$$\boxed{\langle H \rangle} = \underbrace{\frac{\langle p^2 \rangle}{2m}}_{A} + \underbrace{\frac{1}{2} m\omega^2 \langle x^2 \rangle}_{B}$$

$$\boxed{\geq} 2\sqrt{A}\sqrt{B} = 2\,\frac{\langle p^2 \rangle^{1/2}}{(2m)^{1/2}} \cdot \left(\frac{m}{2}\right)^{1/2} \omega \langle x^2 \rangle^{1/2} = \boxed{\Delta p\, \Delta x\, \omega}$$

so $\boxed{\langle H \rangle_{min} =} \Delta p \Delta x\, \omega = \boxed{\frac{1}{2}\hbar\omega}$

2. (a) With $m = \hbar = \omega = 1$, $a = \frac{1}{\sqrt{2}}(x + ip)$ & $a^\dagger = \frac{1}{\sqrt{2}}(x - ip)$,

so $a^\dagger a = \frac{1}{2}(x + ip)(x - ip)$ with $\boxed{p \rightarrow} -i\hbar \frac{d}{dx} = \boxed{-i\frac{d}{dx}}$

and in coordinate representation

$\boxed{a^\dagger a \longrightarrow \frac{1}{2}\left(x - \frac{d}{dx}\right)\left(x + \frac{d}{dx}\right)}$ with $\boxed{a^\dagger a |n\rangle = n|n\rangle}$ and $|n\rangle \rightarrow \psi_n(x)$

$$\boxed{\frac{1}{2}\left(x - \frac{d}{dx}\right)\left(x + \frac{d}{dx}\right)\underbrace{(2x^3 - 3x)e^{-x^2/2}}_{\psi_n(x)}}$$

$$= \frac{1}{2}\left(x - \frac{d}{dx}\right)\left[(2x^4 - 3x^2) + (2\cdot 3x^2 - 3) + (2x^3 - 3x)\left(-\frac{1}{2}\cdot 2x\right)\right]e^{-\frac{x^2}{2}}$$

$$= \frac{1}{2}\left(x - \frac{d}{dx}\right)(6x^2 - 3)e^{-x^2/2} \quad ① \text{ [used in part (b)]}$$

$$= \frac{1}{2}\left[(6x^3 - 3x) - (12x) - (6x^2 - 3)(-x)\right]e^{-x^2/2}$$

$$= \frac{1}{2}(12x^3 - 18x)e^{-x^2/2} = 3(2x^3 - 3x)e^{-x^2/2} = \boxed{3\,\psi_n(x)}$$

so $\boxed{n = 3}$

2.(b) $a|n\rangle = \sqrt{n}\,|n-1\rangle$ with $a \rightarrow \frac{1}{\sqrt{2}}\left(x + \frac{d}{dx}\right)$, $|n\rangle \rightarrow \psi_n(x)$, $|n-1\rangle \rightarrow \psi_{n-1}(x)$

$$\boxed{\psi_{n-1}(x) =} \frac{1}{\sqrt{n}} \cdot \frac{1}{\sqrt{2}}\left(x + \frac{d}{dx}\right)(2x^3 - 3x)\, e^{-x^2/2}$$

$\psi_2(x)$

$n=3 \rightarrow$

$$= \frac{1}{\sqrt{6}}(6x^2 - 3)\, e^{-x^2/2}$$ from ① on first page

$$= \boxed{\sqrt{\frac{3}{2}}\,(2x^2 - 1)\, e^{-x^2/2}}$$ but differing from correct normalization by same factor as $\psi_3(x)$

3. (a) This is an <u>estimate</u>, so <u>solutions will vary</u>.

If we take $\Delta p \Delta x \sim \hbar$, $d \equiv \Delta x$, $x \sim \frac{\Delta x}{2} = \frac{d}{2}$,

$$p \sim \Delta p \sim \frac{\hbar}{\Delta x} = \frac{\hbar}{d}:$$

$$E = \frac{p^2}{2m} + b|x|^n = \frac{1}{2m}\frac{\hbar^2}{d^2} + b\frac{d^n}{2^n}$$

$$0 = \frac{dE}{d(d)} = \frac{\hbar^2}{2m}(-2d^{-3}) + \frac{b}{2^n}(n d^{n-1})$$

$$\Rightarrow d^{n+2} = \frac{\hbar^2}{m} \cdot \frac{2^n}{bn} \quad \text{or} \quad \boxed{d = \left(\frac{2^n}{bn}\frac{\hbar^2}{m}\right)^{\frac{1}{n+2}}}$$

Then
$$\boxed{E \sim} \frac{\hbar^2}{2m}\left(\frac{2^n}{bn}\frac{\hbar^2}{m}\right)^{-\frac{2}{n+2}} + \frac{b}{2^n}\left(\frac{2^n}{bn}\frac{\hbar^2}{m}\right)^{\frac{n}{n+2}}$$

$$= \frac{\hbar^2}{2m}\left(\frac{2^n}{bn}\frac{\hbar^2}{m}\right)^{-\frac{2}{n+2}}\left[1 + \frac{2m}{\hbar^2}\frac{b}{2^n}\left(\frac{2^n}{bn}\frac{\hbar^2}{m}\right)^{\frac{n+2}{n+2}}\right]$$

$$= \boxed{\frac{\hbar^2}{2m}\left(\frac{2^n}{bn}\frac{\hbar^2}{m}\right)^{-\frac{2}{n+2}}\left(1 + \frac{2}{n}\right)}$$

(b) $n = 2$, $b = \frac{1}{2}m\omega^2$:

$$\boxed{E \sim} \frac{\hbar^2}{2m}\left(\frac{4}{m\omega^2}\frac{\hbar^2}{m}\right)^{-\frac{1}{2}}(1+1)$$

$$= \frac{\hbar^{\cancel{2}}}{\cancel{m}}\,\frac{\cancel{m}\omega}{2\cancel{\hbar}}$$

$$= \boxed{\frac{1}{2}\hbar\omega}$$ which atypically is exactly correct

4, (a) $\vec{B} = \vec{\nabla} \times \vec{A}$ and $\vec{E} = -\vec{\nabla}\varphi$ will give you this with various ways of writing the equations

(b) The only coordinate that enters is x, so p_y and p_z commute with H, and their time derivatives are zero by their Heisenberg equations of motion.

(c) Since p_y and p_z commute with H, the eigenstates of H can be taken to also be eigenstates of p_y and p_z.

(d) After careful algebra, one obtains this H with

$$x_0 = \frac{c\,p_y}{q B_0} + \frac{mc^2 E_0}{q B_0^2}$$

(e) $$\langle \vec{v} \rangle = -\frac{q B_0}{mc} \langle x \rangle \hat{y}$$

But with $p_y = 0$,

$$\langle x \rangle = x_0 = \frac{mc^2 E_0}{q B_0^2}$$

so $$\langle \vec{v} \rangle = -\frac{q B_0}{mc} \cdot \frac{mc^2 E_0}{q B_0^2} = -\frac{E_0}{B_0} c \hat{y}.$$

[This answer was given. To get it legitimately, one must find x_0 through a proper calculation.]